$u \in V \quad a(u,v) = f(v) \quad \forall v \in V$

$u_h \in V_h \quad a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$ Galerkin method

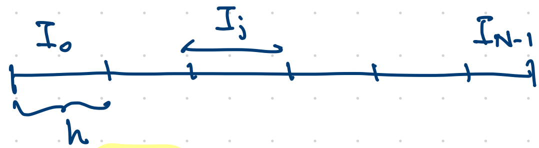
$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$ Quasi-optimality.

Starting now, we construct spaces V_h via polynomials.

Chapter 2 The Finite element method

2.1 Finite element method in 1D

$\Omega = (a,b)$

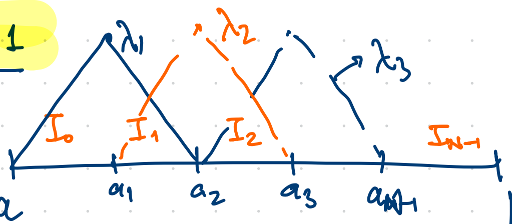


$$V_h = \{ v_h \in V : v_h|_{I_j} \in \mathbb{P}_r(I_j) \quad \forall j = 0 \dots N-1 \}.$$

polynomials of degree r on the interval I_j

$V = H^1_0(\Omega) \quad , \quad H^1(\Omega)$ (functions in V_h must be continuous then)

r=1



$\lambda_j \in V_h \quad (r=1)$

$$\lambda_j = \begin{cases} = 1 & x = a_j \\ = 0 & x = a_i \quad i \neq j \end{cases}$$

$V = H^1_0(\Omega)$

$$\boxed{V_h = \text{span}\{\lambda_1 \dots \lambda_{N+1}\}.}$$

Remark : $\operatorname{span}\{\lambda_1 \dots \lambda_{N-1}\} \subseteq V_h$ because $\lambda_j \in V_h \ \forall j$

$V_h \subseteq \operatorname{span}\{\lambda_1 \dots \lambda_{N-1}\}$? is it ?

$w_h \in V_h$ I try to write it as a linear comb. of $\{\lambda_j\}$.

$w_h(a_j) = c_j$ $\displaystyle\sum_{j=1}^{N-1} c_j \lambda_j(x)$ is equal to w_h.

$$\left(w_h - \sum_{j=1}^{N-1} c_j \lambda_j\right) \in V_h$$

$$\left(w_h - \sum_{j=1}^{N-1} c_j \lambda_j\right)\Bigg|_{x=a_k} = w_h(a_k) - \sum c_j \lambda_j(a_k) = w_h(a_k) - c_k = 0 \quad \forall k$$

I_j

$w_h - \sum c_j \lambda_j = 0$ in every I_j

$\rightarrow w_h \in \operatorname{span}\{\lambda_1 \dots \lambda_{N-1}\}$

definition: degrees of freedom

$$\delta_j(f) = f(a_j) \qquad j = 0 \dots N \qquad (1)$$

definition: the set of degrees of freedom $\{\delta_j\}$ is called $\boxed{\text{UNISOLVENT}}$ in V_h iff

$$\delta_j(v_h) = 0 \ \forall j \implies v_h = 0.$$

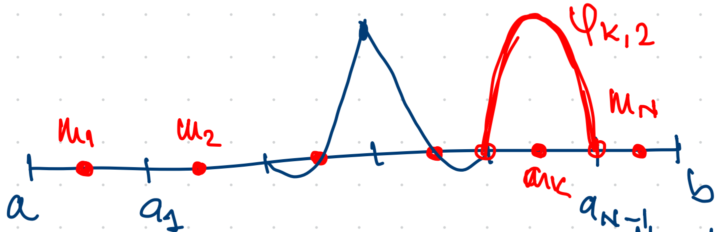
theorem $\{\delta_j\}$ as defined in (1) are unisolvent in V_h.

proof : exercise for you.

$r=2$ $\quad V_h = \{ v_h \in H_0^1(\Omega) : v_h|_{I_j} \in \mathbb{P}_2(I_j) \ \forall j \}.$

- construction of a set of basis functions.

Lagrangian basis (based on point evaluation)



$\varphi_{i,1}$ $\quad i = 1 .. N-1$ $\quad$ attached to vertices a_j

$\varphi_{k,2}$ $\quad k = 1 .. N$ $\quad$ attached to m_k

$$\begin{cases} \varphi_{j,1}(a_i) = \delta_{ij} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \\ \varphi_{j,1}(m_k) = 0 \quad \forall k. \end{cases}$$

$$\begin{cases} \varphi_{k,2}(m_\ell) = \delta_{\ell k} \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases} \\ \varphi_{k,2}(a_j) = 0 \quad \forall j. \end{cases}$$ (bubble)

proposition $\quad V_h = \text{span}\{ \varphi_{j,1}, \varphi_{k,2} \quad \begin{matrix} j = 1 .. N-1 \\ k = 1 .. N \end{matrix} \}.$

the set of degrees of freedom for V_h $(r=2)$

is (2) $\boxed{\delta_\ell(f) = f(x_\ell) \quad x_\ell < \begin{matrix} a_j & \forall j \\ m_k & \forall k. \end{matrix}}$

remark $\{\varphi_{1,1}, \varphi_{2,1} \dots \varphi_{N-1,1}, \varphi_{1,2} \dots \varphi_{N,2}\}$

$= \{\varphi_1, \varphi_2 \dots \varphi_L\} \quad L = 2N-1$

$$\delta_\ell(\varphi_{\ell'}) = \delta_{\ell\ell'} = \begin{cases} 1 & \ell = \ell' \\ 0 & \ell \neq \ell' \end{cases}$$

Proposition (2) is a unisolvent set of degrees of freedom for V_h

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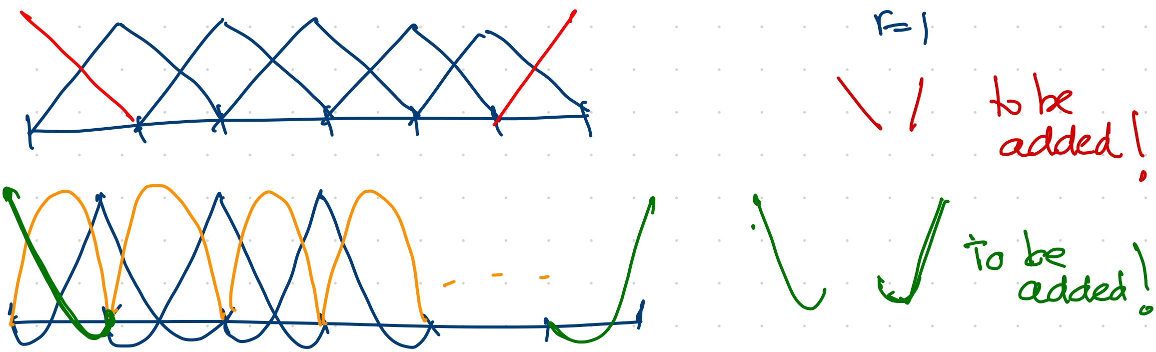
For the space $H^1(\Omega)$



$V_h = \text{span}\{\varphi_1 \dots \varphi_{N_h}\} \quad N_h = \dim(V_h)$

$u_h \in V_h \qquad a(u_h, v_h) = f(v_h) \qquad \forall v_h \in V_h$

this problem becomes a linear system?

$u_h = \sum c_j \varphi_j \qquad\qquad v_h = \varphi_\ell$

$$\sum c_j \, a(\varphi_j, \varphi_\ell) = f(\varphi_\ell) \qquad \forall \ell.$$

$A_{j\ell} = a(\varphi_j, \varphi_\ell)$

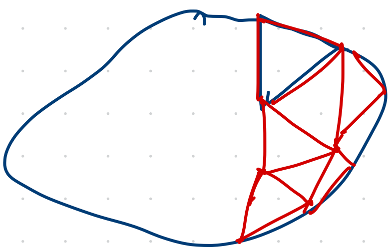
$(\underline{F})_\ell = f(\varphi_\ell)$

$$\boxed{\underline{\underline{A}} \, \underline{c} = \underline{F}}$$

rhs or load

stiffness matrix

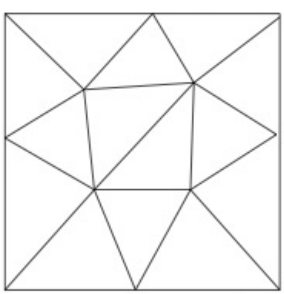
2.2 Finite element in 2D



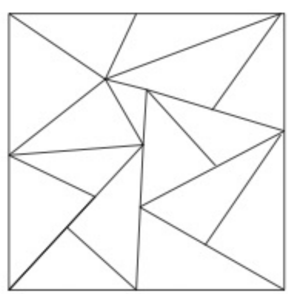
T_h partition of Ω.

- Indeed we partition in triangles, and T_h is called mesh.

(• quadrangles are also possible)



conforming mesh



non-conforming mesh

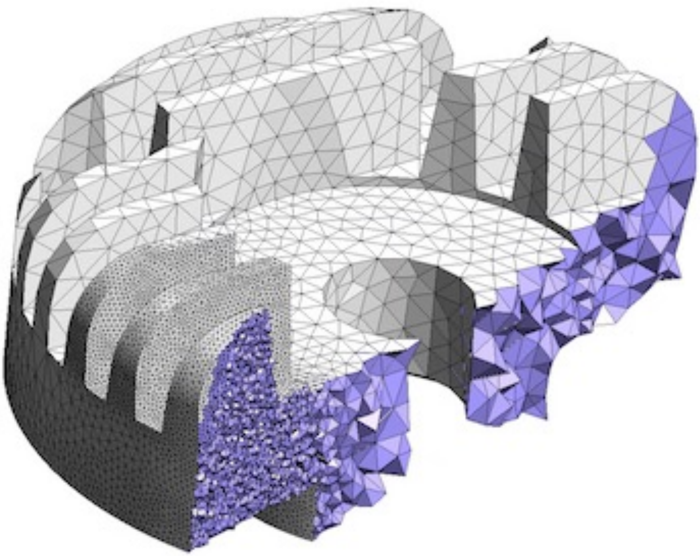
def $\mathcal{T}_h$ is a conforming mesh.

K_i, K_j two elements of $\mathcal{T}_h$

$$K_i \cap K_j = \begin{cases} \emptyset \\ \text{vertex of } \mathcal{T}_h \\ \text{a full edge.} \end{cases}$$

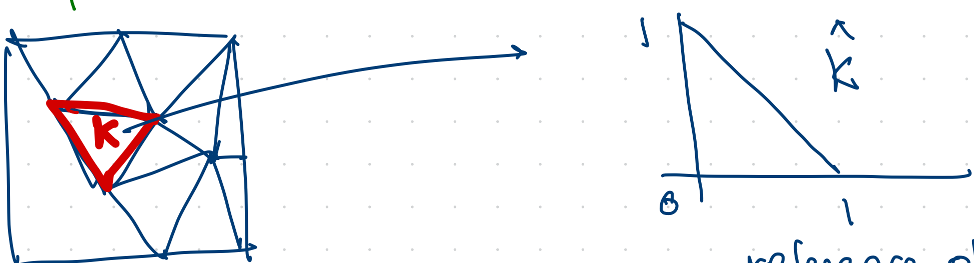
remark:

In 3D $\mathcal{T}_h$ is made of tetrahedra but also quads are possible.



meshing process.

Map to the reference element



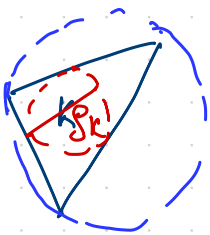
reference element

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ . .

$$\underline{F}_K = \underline{B}_K \hat{\underline{x}} + \underline{b}_K \qquad \underline{b}_K = \underline{a}$$

$$\underline{B}_K = \begin{pmatrix} c_1 - a_1 & b_1 - a_1 \\ c_2 - a_2 & b_2 - a_2 \end{pmatrix}$$



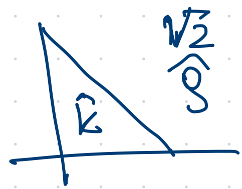
h_K = diameter of K

diameter of the outer circle

ρ_K = diameter of the inner circle

inner diameter of K

$$\|B_K\| \leq \frac{h_K}{\hat{\rho}} \qquad \|B_K^{-1}\| \leq \frac{\sqrt{2}}{\rho_K}$$



$$\det B_K = \frac{1}{2} |K|$$

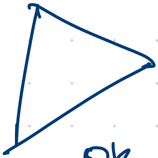
area of K.

exercise (in the exercise session)

<u>definition</u> We say T_h is <u>regular</u> if

$\exists\ C_R \quad R > 0$ "independent of h"

$\forall K \in T_h \qquad h_K \leq C_R \rho_K$



ok

not ok

not ok.

$$\mathcal{T}_h \qquad h = \max_{K\in\mathcal{T}_h} h_K$$

Construction of finite element spaces

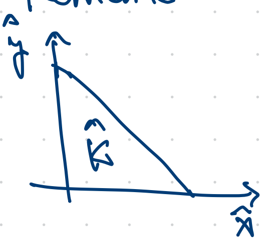
$\Omega \subseteq \mathbb{R}^2$ $\qquad$ $\mathcal{T}_h$ a family of meshes parametrized in h.

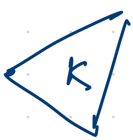
$$V_h = \left\{ v_h \in \begin{matrix} H^1_0(\Omega) \\ H^1(\Omega) \end{matrix} : u_h|_K \in \mathbb{P}_r(K) \quad \forall K\in\mathcal{T}_h \right\}$$

polynomials of degree r on K.

$\boxed{r=1}$ for the time being.

Remark $\qquad$ $\mathbb{P}_1(K)$





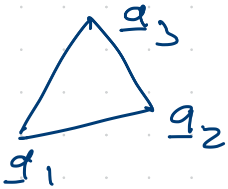
$$\mathbb{P}_1(\hat{K}) = \text{span}\{\underbrace{\hat{x}, \hat{y}, 1-\hat{x}-\hat{y}}_{\text{baricentric coordinates of } \hat{K}}\}$$

$$\hat{\varphi}_1 = \hat{x}, \quad \hat{\varphi}_2 = \hat{y}, \quad \hat{\varphi}_3 = 1-\hat{x}-\hat{y}$$

Basis functions for $\mathbb{P}_1(K)$ are constructed as:

$$\varphi_j(F_K \hat{x}) = \hat{\varphi}_j(\hat{x})$$

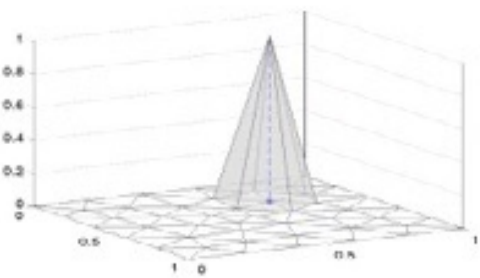
$\varphi_1, \varphi_2, \varphi_3$ are the barycentric coordinates of K

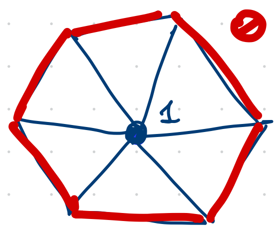


$$\varphi_i(\underline{a}_j) = \delta_{ij}.$$

<u>Theorem</u> $V_h = \text{span}\{\varphi_v,\ v \text{ internal vertex of } \mathcal{T}_h\}$

where $\varphi_v \in V_h$, $\varphi_v = \begin{cases} 1 & \text{on } v \\ 0 & \text{on all other vertices of } \mathcal{T}_h \end{cases}$





$\varphi_v|_K$ = barycentric coordinate of in K for the vertex v, $\forall K$ that has v as a vertex.

<u>proof of the theorem</u>

- $\text{span}\{\varphi_v,\ v \text{ internal vertex}\} \subseteq V_h$ easy.
- $V_h \subseteq \text{span}\{\varphi \ldots\}$ should be proved (next time)

$$\dim(V_h) = N_h = \#\text{ internal vertices of } V_h.$$

Degrees of freedom $\qquad \delta_j(f) = f(\underline{v}_j)$

$\underline{v}_j$ = the j vertex of T_h.

D.of. are used to define "interpolation" operator.

$f \in C^0(\Omega)$ $\qquad \Pi_1 f(x) = \sum_{j=1}^{N_h} \underbrace{f(v_j)}_{\delta_j(f)} \cdot \varphi_{v_j}(x)$

degree r=1.

$\Pi_1 f$ is called interpolation of f on V_h.